LAST TIME - DOUBLE RIEMANN SUMS

of a function $z = f(x,y)$

over a rectangle $R = [a,b] \times [c,d]$:

$$\text{RIEMANN SUM} = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A_{ij},$$

THEN we define the number

$$\iint_R f(x,y)\, dA = \lim_{\substack{m \to \infty \\ n \to \infty}} \left(\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta A_{ij} \right)$$

How we find this number?

By USING ITERATED Integrals

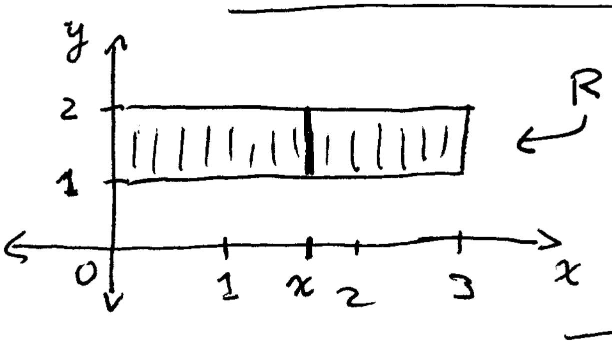
FOR EXAMPLE: Let $R = [0,3] \times [1,2]$
$\qquad\qquad\qquad\qquad x\text{'s} \qquad y\text{'s}$

Let $f(x,y) = y^2 + 8xy^3 + x^3$.

How do we determine the number

$$\iint_R (y^2 + 8xy^3 + x^3)\, dA \ ?$$

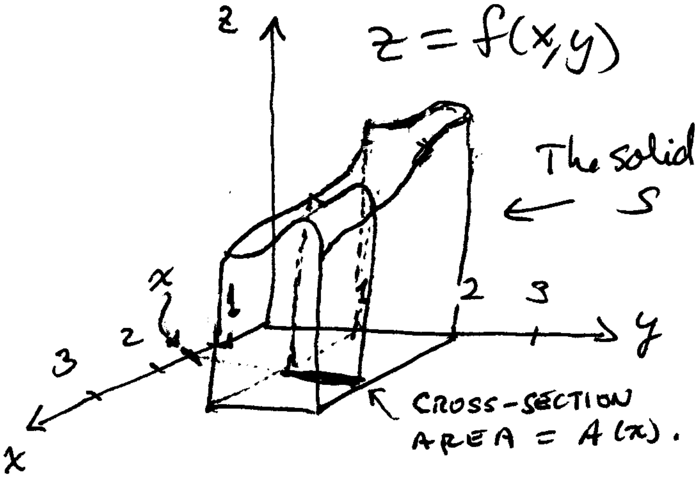
Here, it is the Volume of the solid region above Rectangle R and below the surface graph.



Use Iterated Integrals:

Let x be fixed at a value in $[a,b]$, $a \le x \le b$, and, while x is held constant, let y vary in $[c,d]$, $c \le y \le d$.

Then $g_x(y) = f(x,y)$ defines a function of y for each x, $a \le x \le b$.



Here, for x with $0 \le x \le 3$, define $g_x(y) = y^2 + 8xy^3 + x^3$, that is, $g_x(y) = f(x,y)$.

Here, $g_0(y) = y^2$

$g_1(y) = y^2 + 8y^3 + 1$

$g_{1.5}(y) = y^2 + 12y^3 + 3.375$

$g_2(y) = y^2 + 16y^3 + 8$

$g_3(y) = y^2 + 24y^3 + 27$

Fix x at any number with $a = 0 \le x \le 3 = b$, and define the number $A(x)$ as follows:

$$A(x) = \int_1^2 (y^2 + 8xy^3 + x^3)\,dy = \int_c^d g_x(y)\,dy$$

$$= \left(\tfrac{1}{3}y^3 + 2xy^4 + x^3 y\right)\Big|_{y=1}^{y=2}$$

$$= \left(\tfrac{8}{3} + 32x + 2x^3\right) - \left(\tfrac{1}{3} + 2x + x^3\right)$$

$$A(x) = x^3 + 30x + \tfrac{7}{3}, \qquad 0 \le x \le 3.$$

$A(x)$ is here the cross-sectional area of the solid S cut by the vertical plane through $(x,0,0)$ and perpendicular to the x-axis.

The Volume V OF THE SOLID S is determined by

$$V = \int_0^3 A(x)\,dx = \int_0^3 \left(x^3 + 30x + \tfrac{7}{3}\right)dx = 162.25$$

Also, $$V = \iint_R (y^2 + 8xy^3 + x^3)\,dA = 162.25$$

By Fubini's Theorem (p. 984)

$$\iint_{R=[0,3]\times[1,2]} (y^2+8xy^3+x^3)\,dA = \underbrace{\int_{x=0}^{3}\Bigg(\overbrace{\int_{y=1}^{2}\underbrace{(y^2+8xy^3+x^3)\,dy}_{\text{Integrate w.r.t. } y}}^{A(x)}\Bigg)dx}_{=162.25}$$

AN ITERATED INTEGRAL →

Fix x

Also, by Fubini's Theorem, you can

SWITCH THE ORDER OF INTEGRATION:

$$\iint_R (y^2+8xy^3+x^3)\,dA = \int_{y=1}^{2}\Bigg(\underbrace{\int_{x=0}^{3}(y^2+8xy^3+x^3)\,dx}_{A(y)}\Bigg)dy$$

Fix y

$= 162.25$

Problem: Determine the volume V of the solid S below the surface graph of $z = 16 - x^2 - y^2$ and above the rectangle $[0,1]\times[0,1]$.

SOLUTION: $V = \iint_R (16-x^2-y^2)\,dA$, so,

$$V = \int_0^1\int_0^1 (16-x^2-y^2)\,dy\,dx$$

$$= \int_0^1\left(\int_0^1 (16-x^2-y^2)\,dy\right)dx$$

Fix x and work from the inside out!

$$= \int_0^1\left(\left[16y - x^2y - \tfrac{1}{3}y^3\right]_{y=0}^{y=1}\right)dx$$

$$V = \int_0^1 \left(\left(16 - x^2 - \tfrac{1}{3}\right) - (0) \right) dx$$

$$= \int_0^1 \left(\underset{\nwarrow\, 48/3}{16} - x^2 - \tfrac{1}{3}\right) dx = \int_0^1 \left(47/3 - x^2\right) dx$$

$$= \left[\tfrac{47}{3} x - \tfrac{1}{3} x^3 \right]_0^1 = \left(\tfrac{47}{3} - \tfrac{1}{3} \right) - (0 - 0)$$

$$V = \frac{46}{3} = 15\tfrac{1}{3} \text{ cubic units.}$$

Look ahead and be careful which order you choose to do the integrating in.

If you end up with a difficult integral, consider switching the order of integration.

Ex: $R = [1,2] \times [0, \pi]$.

Determine $\iint_R y \sin(xy)\, dA$.

You have two choices - Which order should you choose?

$$\int_1^2 \int_0^\pi \underbrace{y \sin(xy)}\, dy\, dx \quad \text{OR} \quad \int_0^\pi \int_1^2 \underbrace{y \sin(xy)}\, dx\, dy$$

A PRODUCT OF TWO FUNCTIONS OF y to INTEGRATE "dy".

↑

MUST USE INTEGRATION BY PARTS HERE

A PRODUCT OF A CONSTANT TIMES A FUNCTION of x to integrate "dx"

↑

You CAN USE u-SUBSTITUTION HERE.

CHOOSE THIS ONE!

Compare these functions:

$$f_1(x,y) = x^4 \sin y \quad \text{and} \quad f_2(x,y) = x^4 \sin(xy)$$

$$f_1(x,y) = \underset{x^4}{g(x)}\,\underset{\sin y}{h(y)}$$

NOT HERE!

When $f(x,y) = g(x)\,h(y)$,

$$\int_a^b \int_c^d (g(x)\,h(y))\,dy\,dx = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$$

because

$$\int_a^b \left(\int_c^d g(x)\,h(y)\,dy\right) dx$$

$$= \int_a^b \left(g(x)\int_c^d h(y)\,dy\right) dx$$

$$= \int_a^b \left(g(x)\left(\int_c^d h(y)\,dy\right)\right) dx$$

$$= \left(\int_c^d h(y)\,dy\right)\int_a^b g(x)\,dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

So, $\int_0^1 \int_0^{\pi/2} x^4 \sin y\,dy\,dx = \left(\int_0^1 x^4\,dx\right)\left(\int_0^{\pi/2} \sin y\right)dy = \left(\frac{1}{5}\right)(1)$

$$= \frac{1}{5}$$